Elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and vertex operators

2008 J. Phys. A: Math. Theor. 41 194012

(http://iopscience.iop.org/1751-8121/41/19/194012)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 194012 (12pp) doi:10.1088/1751-8113/41/19/194012

Elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and vertex operators

Hitoshi Konno

Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima 739-8521, Japan

E-mail: konno@mis.hiroshima-u.ac.jp

Received 7 November 2007, in final form 25 February 2008
Published 29 April 2008
Online at stacks.iop.org/JPhysA/41/194012

Abstract

Introducing an H-Hopf algebroid structure into $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, we investigate the vertex operators of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ defined as intertwining operators of infinite dimensional $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ modules. We show that the vertex operators coincide with the previous results obtained indirectly by using the quasi-Hopf algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$. This shows a consistency of our H-Hopf algebroid structure even in the case with a nonzero central element.

PACS numbers: 02.20.Uw, 02.90.+p, 05.50.+q
Mathematics Subject Classification: 16W30, 17B37, 17B67, 17B69, 81R50

1. The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

In this section, we review a definition of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and its RLL formulation following [1, 2].

1.1. Definition of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ was introduced in [1] as an elliptic analogue of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ in the Drinfeld realization. It was soon realized that $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is isomorphic to the tensor product of $U_q(\widehat{\mathfrak{sl}}_2)$ and a Heisenberg algebra $\{P, \mathrm{e}^Q\}$ [2]. We here define $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ along the latter observation.

Let us fix a complex number q such that $q \neq 0$, $|q| < 1$.

Definition 1.1 [3]**.** *For a field $\mathbb{K}$, the quantum affine algebra $\mathbb{K}[U_q(\widehat{\mathfrak{sl}}_2)]$ in the Drinfeld realization is an associative algebra over $\mathbb{K}$ generated by the Drinfeld generators $a_n (n \in \mathbb{Z}_{\neq 0})$, $x_n^\pm (n \in \mathbb{Z})$, h, c, d. The defining relations are given as follows:*

$$c: \ \textit{central},$$
$$[h, d] = 0, \qquad [d, a_n] = n a_n, \qquad \left[d, x_n^\pm\right] = n x_n^\pm,$$
$$[h, a_n] = 0, \qquad [h, x^\pm(z)] = \pm 2 x^\pm(z),$$

$$[a_n, a_m] = \frac{[2n]_q[cn]_q}{n} q^{-c|n|} \delta_{n+m,0},$$

$$[a_n, x^+(z)] = \frac{[2n]_q}{n} q^{-c|n|} z^n x^+(z),$$

$$[a_n, x^-(z)] = -\frac{[2n]_q}{n} z^n x^-(z),$$

$$(z - q^{\pm 2}w)x^\pm(z)x^\pm(w) = (q^{\pm 2}z - w)x^\pm(w)x^\pm(z),$$

$$[x^+(z), x^-(w)] = \frac{1}{q - q^{-1}}\left(\delta\left(q^{-c}\frac{z}{w}\right)\psi(q^{c/2}w) - \delta\left(q^{c}\frac{z}{w}\right)\varphi(q^{-c/2}w)\right),$$

where $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ *and*

$$x^\pm(z) = \sum_{n\in\mathbb{Z}} x^\pm_n z^{-n},$$

$$\psi(q^{c/2}z) = q^h \exp\left((q - q^{-1})\sum_{n>0} a_n z^{-n}\right),$$

$$\varphi(q^{-c/2}z) = q^{-h} \exp\left(-(q - q^{-1})\sum_{n>0} a_{-n} z^{n}\right).$$

Let r be a complex parameter. We set $r^* = r - c$, $p = q^{2r}$ and $p^* = q^{2r^*}$. We define the Jacobi theta functions $[u]$ and $[u]^*$ by

$$[u] = \frac{q^{u^2/r-u}}{(p;\,p)^3_\infty}\Theta_p(q^{2u}), \qquad [u]^* = \frac{q^{u^2/r^*-u}}{(p^*;\,p^*)^3_\infty}\Theta_{p^*}(q^{2u}),$$

where

$$\Theta_p(z) = (z;\,p)_\infty(p/z;\,p)_\infty(p;\,p)_\infty,$$

$$(z;\,p_1, p_2, \ldots, p_m)_\infty = \prod_{n_1,n_2,\ldots,n_m=0}^{\infty}\left(1 - zp_1^{n_1}p_2^{n_2}\cdots p_m^{n_m}\right).$$

Setting $p = \mathrm{e}^{-2\pi \mathrm{i}/\tau}$, $[u]$ satisfies the quasi-periodicity $[u + r] = -[u]$, $[u + r\tau] = \mathrm{e}^{-\pi\mathrm{i}(2u/r+\tau)}[u]$.

We denote by $\{P, \mathrm{e}^Q\}$ a Heisenberg algebra commuting with $\mathbb{C}[U_q(\widehat{\mathfrak{sl}}_2)]$ and satisfying

$$[P, \mathrm{e}^Q] = -\mathrm{e}^Q. \tag{1.1}$$

We take the realization $Q = \frac{\partial}{\partial P}$. We set $H = \mathbb{C}P \oplus \mathbb{C}r^*$ and $H^* = \mathbb{C}Q \oplus \mathbb{C}\frac{\partial}{\partial r^*}$ with the pairing $\langle\,,\rangle$

$$\langle Q, P\rangle = 1 = \left\langle\frac{\partial}{\partial r^*}, r^*\right\rangle,$$

the others are zero.

We also consider the Abelian group $\bar{H}^* = \mathbb{Z}Q$. We denote by $\mathbb{C}[\bar{H}^*]$ the group algebra over $\mathbb{C}$ of $\bar{H}^*$, and by e^α the element of $\mathbb{C}[\bar{H}^*]$ corresponding to $\alpha \in \bar{H}^*$. These e^α satisfy $\mathrm{e}^\alpha\mathrm{e}^\beta = \mathrm{e}^{\alpha+\beta}$ and $(\mathrm{e}^\alpha)^{-1} = \mathrm{e}^{-\alpha}$. In particular, $\mathrm{e}^0 = 1$ is the identity element.

Now we take the power series field $\mathbb{F} = \mathbb{C}((P, r^*))$ as $\mathbb{K}$ and consider the semi-direct product $\mathbb{C}$-algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2) = \mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)] \otimes_{\mathbb{C}} \mathbb{C}[\bar{H}^*]$ of $\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)]$ and $\mathbb{C}[H^*]$, whose multiplication is defined by

$$(f(P, r^*)a \otimes \mathrm{e}^\alpha)\cdot(g(P, r^*)b \otimes \mathrm{e}^\beta) = f(P, r^*)g(P + \langle\alpha, P\rangle, r^*)ab \otimes \mathrm{e}^{\alpha+\beta},$$

$$a, b \in \mathbb{C}[U_q(\widehat{\mathfrak{sl}}_2)],\ f(P, r^*), g(P, r^*) \in \mathbb{F}, \alpha, \beta \in \bar{H}^*.$$

Let us consider the following generating functions:

$$u^+(z,p)=\exp\left(\sum_{n>0}\frac{1}{[r^*n]_q}a_{-n}(q^rz)^n\right),\qquad u^-(z,p)=\exp\left(-\sum_{n>0}\frac{1}{[rn]_q}a_n(q^{-r}z)^{-n}\right).$$

We define an automorphism ϕ_r of $\mathbb{C}[U_q(\widehat{\mathfrak{sl}}_2)]$ by

$$\begin{aligned}
&c\mapsto c,\qquad h\mapsto h,\qquad d\mapsto d,\\
&x^+(z)\mapsto u^+(z,p)x^+(z),\qquad x^-(z)\mapsto x^-(z)u^-(z,p),\\
&\psi(z)\mapsto u^+(q^{c/2}z,p)\psi(z)u^-(q^{-c/2}z,p),\\
&\varphi(z)\mapsto u^+(q^{-c/2}z,p)\varphi(z)u^-(q^{c/2}z,p).
\end{aligned}$$

Definition 1.2. *We define $E(u), F(u), K(u)\in U_{q,p}(\widehat{\mathfrak{sl}}_2)[[u]]$ and $\hat{d}$ by the following formulae:*

$$\begin{aligned}
E(u)&=\phi_r(x^+(z))\,\mathrm{e}^{2Q}z^{-(P-1)/r^*},\\
F(u)&=\phi_r(x^-(z))z^{(P+h-1)/r},\\
K(u)&=\exp\left(\sum_{n>0}\frac{[n]_q}{[2n]_q[r^*n]_q}a_{-n}(q^cz)^n\right)\exp\left(-\sum_{n>0}\frac{[n]_q}{[2n]_q[rn]_q}a_nz^{-n}\right)\\
&\quad\times\mathrm{e}^{Q}z^{-c(2P-1)/4rr^*+h/2r},\\
\hat{d}&=d-\frac{1}{4r^*}(P-1)(P+1)+\frac{1}{4r}(P+h-1)(P+h+1),
\end{aligned}$$

where we set $z=q^{2u}$. We call $E(u), F(u), K(u)$ the elliptic currents.

In fact, from definition 1.1 and (1.1), we can derive the following relations.

Proposition 1.3.

$$\begin{aligned}
&c:\ \text{central},\\
&[h,a_n]=0,\qquad [h,E(u)]=2E(u),\qquad [h,F(u)]=-2F(u),\\
&[\hat{d},h]=0,\qquad [\hat{d},a_n]=na_n,\\
&[\hat{d},E(u)]=\left(-z\frac{\partial}{\partial z}-\frac{1}{r^*}\right)E(u),\qquad [\hat{d},F(u)]=\left(-z\frac{\partial}{\partial z}-\frac{1}{r}\right)F(u),\\
&[a_n,a_m]=\frac{[2n]_q[cn]_q}{n}q^{-c|n|}\delta_{n+m,0},\\
&[a_n,E(u)]=\frac{[2n]_q}{n}q^{-c|n|}z^nE(u),\\
&[a_n,F(u)]=-\frac{[2n]_q}{n}z^nF(u),\\
&E(u)E(v)=\frac{[u-v+1]^*}{[u-v-1]^*}E(v)E(u),\\
&F(u)F(v)=\frac{[u-v-1]}{[u-v+1]}F(v)F(u),\\
&[E(u),F(v)]=\frac{1}{q-q^{-1}}\left(\delta\left(q^{-c}\frac{z}{w}\right)H^+(q^{c/2}w)-\delta\left(q^c\frac{z}{w}\right)H^-(q^{-c/2}w)\right),
\end{aligned}$$

where $z=q^{2u}$, $w=q^{2v}$,

$$\begin{aligned}
&H^{\pm}(z)=\kappa K\left(u\pm\frac{1}{2}\left(r-\frac{c}{2}\right)+\frac{1}{2}\right)K\left(u\pm\frac{1}{2}\left(r-\frac{c}{2}\right)-\frac{1}{2}\right),\\
&\kappa=\lim_{z\to q^{-2}}\frac{\xi(z;p^*,q)}{\xi(z;p,q)},\qquad \xi(z;p,q)=\frac{(q^2z;p,q^4)_\infty(pq^2z;p,q^4)_\infty}{(q^4z;p,q^4)_\infty(pz;p,q^4)_\infty}.
\end{aligned}$$

In particular, we have the following relations which, together with the last three relations in the above, appeared in [1].

Proposition 1.4.

$$K(u)K(v) = \rho(u-v)K(v)K(u),$$
$$K(u)E(v) = \frac{\left[u-v+\frac{1-r^*}{2}\right]^*}{\left[u-v-\frac{1+r^*}{2}\right]^*}E(v)K(u),$$
$$K(u)F(v) = \frac{\left[u-v-\frac{1+r}{2}\right]}{\left[u-v+\frac{1-r}{2}\right]}F(v)K(u),$$
$$H^+(u)H^-(v) = \frac{\left[u-v-1-\frac{c}{2}\right]}{\left[u-v+1-\frac{c}{2}\right]}\frac{\left[u-v+1+\frac{c}{2}\right]^*}{\left[u-v-1+\frac{c}{2}\right]^*}H^-(v)H^+(u),$$
$$H^\pm(u)H^\pm(v) = \frac{[u-v-1]}{[u-v+1]}\frac{[u-v+1]^*}{[u-v-1]^*}H^\pm(v)H^\pm(u),$$

where

$$\rho(u) = \frac{\rho^{+*}(u)}{\rho^+(u)}, \qquad \rho^+(u) = z^{1/2r}\frac{\{pq^2z\}^2}{\{pz\}\{pq^4z\}}\frac{\{z^{-1}\}\{q^4z^{-1}\}}{\{q^2z^{-1}\}^2}, \qquad \{z\} = (z;p,q^4)_\infty,$$
$$\rho^{+*}(u) = \rho^+(u)|_{r\to r^*}.$$

Definition 1.5. *We call a set* $(\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)]\otimes_{\mathbb{C}}\mathbb{C}[\bar{H}^*],\phi_r)$ *the elliptic algebra* $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

The following relations are also useful.

Proposition 1.6.

$$[K(u),P] = K(u), \qquad [E(u),P] = 2E(u), \qquad [F(u),P] = 0,$$
$$[K(u),P+h] = K(u), \qquad [E(u),P+h] = 0, \qquad [F(u),P+h] = 2F(u).$$

1.2. The RLL relation for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

We next summarize the RLL relation for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [2]. In the following section, the L operator is used to discuss the H-Hopf algebroid structure of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

Let us define the half currents in the following way.

Definition 1.7.

$$K^+(u) = K\left(u+\frac{r+1}{2}\right),$$
$$E^+(u) = a^*\oint_{C^*}E(u')\frac{[u-u'+c/2-P+1]^*[1]^*}{[u-u'+c/2]^*[P-1]^*}\frac{dz'}{2\pi iz'},$$
$$F^+(u) = a\oint_C F(u')\frac{[u-u'+P+h-1][1]}{[u-u'][P+h-1]}\frac{dz'}{2\pi iz'}.$$

Here the contours are chosen such that

$$C^*: |p^*q^cz| < |z'| < |q^cz|, \qquad C: |pz| < |z'| < |z|,$$

and the constants a, a^* *are chosen to satisfy* $\frac{a^*a[1]^*\kappa}{q-q^{-1}} = 1$.

Definition 1.8. *We define the operator* $\widehat{L}^+(u) \in \mathrm{End}_{\mathbb{C}} V \otimes U_{q,p}\big(\widehat{\mathfrak{sl}}_2\big)$ *with* $V \cong \mathbb{C}^2$, *by*

$$\widehat{L}^+(u) = \begin{pmatrix} 1 & F^+(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K^+(u-1) & 0 \\ 0 & K^+(u)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ E^+(u) & 1 \end{pmatrix}.$$

Proposition 1.9. *The operator* $\widehat{L}^+(u)$ *satisfies the following* RLL *relation:*

$$R^{+(12)}(u_1-u_2, P+h)\widehat{L}^{+(1)}(u_1)\widehat{L}^{+(2)}(u_2) = \widehat{L}^{+(2)}(u_2)\widehat{L}^{+(1)}(u_1)R^{+*(12)}(u_1-u_2, P), \qquad (1.2)$$

where $R^+(u, P+h)$ *and* $R^{+*}(u, P) = R^+(u, P)|_{r\to r^*}$ *denote the elliptic dynamical R matrices given by*

$$R^+(u,s) = \rho^+(u) \begin{pmatrix} 1 & & & \\ & b(u,s) & c(u,s) & \\ & \bar{c}(u,s) & \bar{b}(u,s) & \\ & & & 1 \end{pmatrix}, \qquad (1.3)$$

with

$$b(u,s) = \frac{[s+1][s-1]}{[s]^2}\frac{[u]}{[1+u]}, \qquad c(u,s) = \frac{[1]}{[s]}\frac{[s+u]}{[1+u]},$$
$$\bar{c}(u,s) = \frac{[1]}{[s]}\frac{[s-u]}{[1+u]}, \qquad \bar{b}(u,s) = \frac{[u]}{[1+u]}.$$

Note that if we set $L^+(u,P) = \widehat{L}^+(u)\,\mathrm{e}^{-h\otimes Q}$, $L^+(u,P)$ is independent of Q and satisfies the dynamical RLL relation [2] characterizing the quasi-Hopf algebra $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ [4]. Moreover, with the parametrization $\lambda = (r^*+2)\Lambda_0 + (P+1)\bar{\Lambda}_1$, where Λ_0, $\Lambda_0 + \bar{\Lambda}_1$ are the fundamental weights of $\widehat{\mathfrak{sl}}_2$, $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ is isomorphic to $\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)]$, as an associative algebra. These two facts lead to the isomorphism $U_{q,p}(\widehat{\mathfrak{sl}}_2) \cong \mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2) \otimes_{\mathbb{C}} \mathbb{C}[\bar{H}^*]$ as a semi-direct product $\mathbb{C}$-algebra. However, this semi-direct product breaks down the quasi-Hopf algebra structure, so that $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is not a quasi-Hopf algebra. In the following section, we show that a relevant co-algebra structure of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is the H-Hopf algebroid.

Note also that the $c = 0$ case of the dynamical RLL relation for $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ coincides with the one studied by Felder [5, 6], whereas the $c = 0$ case of (1.2) coincides with the RLL relation studied in [7–9] for the trigonometric R and in [10] for the elliptic R.

2. *H*-Hopf algebroid structure of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

In this section, we introduce an H-Hopf algebroid structure into $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. The detailed discussion will be published elsewhere [11]. We follow the definition of H-Hopf algebroid given in [7–10] with a modification which makes it applicable in the case with nonzero central element.

Let $\bar{\mathfrak{h}} = \mathbb{C}h$ be the Cartan subalgebra, α_1 the simple root and $\bar{\Lambda}_1$ be the fundamental weight of $\mathfrak{sl}_2$. We set $\mathcal{Q} = \mathbb{Z}\alpha_1$ and $\bar{\mathfrak{h}}^* = \mathbb{C}\bar{\Lambda}_1$. Let us use the same symbol $\langle , \rangle$ to denote the standard paring of $\bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}^*$. Using the isomorphism $\phi : \mathcal{Q} \to \bar{H}^*$ by $n\alpha_1 \mapsto nQ$, we define the $\bar{H}^*$-bigrading structure of $U_{q,p} = U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by

$$U_{q,p} = \bigoplus_{\alpha,\beta\in\bar{H}^*} (U_{q,p})_{\alpha\beta},$$
$$(U_{q,p})_{\alpha\beta} = \left\{ x \in U_{q,p} \;\middle|\; \begin{matrix} q^h x q^{-h} = q^{\langle\bar{\alpha},h\rangle}x, \alpha = \phi(\bar{\alpha}) + \beta \\ q^P x q^{-P} = q^{\langle\beta,P\rangle}x \end{matrix} \right\}. \qquad (2.1)$$

Noting $\langle\bar{\alpha}, h\rangle = \langle\phi(\bar{\alpha}), P\rangle$, we have $q^{P+h}xq^{-(P+h)} = q^{\langle\alpha,P\rangle}x$ for $x \in (U_{q,p})_{\alpha\beta}$.

We regard $\widehat{f} = f(P, r^*) \in \mathbb{F}$ as a meromorphic function on H^* by

$$\widehat{f}(\mu) = f(\langle \mu, P\rangle, \langle \mu, r^*\rangle), \qquad \mu \in H^*$$

and consider the field of meromorphic functions M_{H^*} on H^* given by

$$M_{H^*} = \{\widehat{f} : H^* \to \mathbb{C} | \widehat{f} = f(P, r^*) \in \mathbb{F}\}.$$

We define two embeddings (the left and right moment maps) $\mu_l, \mu_r : M_{H^*} \to (U_{q,p})_{00}$ by

$$\mu_l(\widehat{f}) = f(P + h, r^* + c), \qquad \mu_r(\widehat{f}) = f(P, r^*). \tag{2.2}$$

From (2.1), one finds for $x \in (U_{q,p})_{\alpha\beta}$

$$\mu_l(\widehat{f})x = f(P + h, r^* + c)x = xf(P + h + \langle \alpha, P\rangle, r^* + c) = x\mu_l(T_\alpha \widehat{f}),$$
$$\mu_r(\widehat{f})x = f(P, r^*)x = xf(P + \langle \beta, P\rangle, r^*) = x\mu_r(T_\beta \widehat{f}),$$

where we regard $T_\alpha = \mathrm{e}^\alpha \in \mathbb{C}[\bar{H}^*]$ as a shift operator $M_{H^*} \to M_{H^*}$

$$(T_\alpha \widehat{f}) = \mathrm{e}^\alpha f(P, r^*)\,\mathrm{e}^{-\alpha} = f(P + \langle \alpha, P\rangle, r^*).$$

Hereafter, we abbreviate $f(P + h, r^* + c)$ and $f(P, r^*)$ as $f(P + h)$ and $f^*(P)$, respectively.

Then equipped with the bigrading structure (2.1) and two moment maps (2.2), the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is an H-algebra [7, 8].

In addition, we need the H-algebra $\mathcal{D}$ of the shift operators given by

$$\mathcal{D} = \left\{ \sum_i \widehat{f}_i T_{\alpha_i} \middle| \widehat{f}_i \in M_{H^*}, \alpha_i \in \bar{H}^* \right\},$$
$$(\mathcal{D})_{\alpha\alpha} = \{\widehat{f} T_{-\alpha}\}, \qquad (\mathcal{D})_{\alpha\beta} = 0 \quad \alpha \neq \beta,$$
$$\mu_l^{\mathcal{D}}(\widehat{f}) = \mu_r^{\mathcal{D}}(\widehat{f}) = \widehat{f} T_0, \qquad \widehat{f} \in M_{H^*}.$$

Let A and B be two H-algebras, $U_{q,p}$ or $\mathcal{D}$. The tensor product $A \widetilde{\otimes} B$ is the bigraded vector space with

$$(A \widetilde{\otimes} B)_{\alpha\beta} = \bigoplus_{\gamma \in \bar{H}^*} (A_{\alpha\gamma} \otimes_{M_{H^*}} B_{\gamma\beta}),$$

where $\otimes_{M_{H^*}}$ denotes the usual tensor product modulo the following relations:

$$\mu_r^A(\widehat{f})a \otimes b = a \otimes \mu_l^B(\widehat{f})b, \qquad a \in A, \quad b \in B. \tag{2.3}$$

Then the tensor product $A \widetilde{\otimes} B$ is again an H-algebra with the multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$ and the moment maps

$$\mu_l^{A\widetilde{\otimes} B} = \mu_l^A \otimes 1, \qquad \mu_r^{A\widetilde{\otimes} B} = 1 \otimes \mu_r^B.$$

Note that we have the H-algebra isomorphism $U_{q,p} \widetilde{\otimes} \mathcal{D} \cong U_{q,p} \cong \mathcal{D} \widetilde{\otimes} U_{q,p}$ by $x \widetilde{\otimes} T_{-\beta} = x = T_{-\alpha} \widetilde{\otimes} x$ for $x \in (U_{q,p})_{\alpha\beta}$.

Now let us define an H-Hopf algebroid structure on $U_{q,p}$ as its co-algebra structure. For this purpose, it is convenient to use the L operator $\widehat{L}^+(u)$. We shall write the entries of $\widehat{L}^+(u)$ as

$$\widehat{L}^+(u) = \begin{pmatrix} \widehat{L}^+_{++}(u) & \widehat{L}^+_{+-}(u) \\ \widehat{L}^+_{-+}(u) & \widehat{L}^+_{--}(u) \end{pmatrix}.$$

From proposition 1.6 and definition 1.8, one finds

$$\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u) \in (U_{q,p})_{-\varepsilon_1 Q, -\varepsilon_2 Q}.$$

It is also easy to check the relations

$$f(P+h)\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u) = \widehat{L}^+_{\varepsilon_1\varepsilon_2}(u) f(P+h-\varepsilon_1),$$
$$f^*(P)\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u) = \widehat{L}^+_{\varepsilon_1\varepsilon_2}(u) f^*(P-\varepsilon_2).$$

Definition 2.1. *We define H-algebra homomorphisms, $\varepsilon : U_{q,p} \to \mathcal{D}$ and $\Delta : U_{q,p} \to U_{q,p} \widetilde{\otimes} U_{q,p}$ by*

$$\varepsilon\big(\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u)\big) = \delta_{\varepsilon_1,\varepsilon_2} T_{-\varepsilon_2 Q}, \qquad \varepsilon(\mathrm{e}^Q) = \mathrm{e}^Q,$$
$$\varepsilon(\mu_l(\widehat{f})) = \varepsilon(\mu_r(\widehat{f})) = \widehat{f} T_0,$$
$$\Delta\big(\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u)\big) = \sum_{\varepsilon'} \widehat{L}^+_{\varepsilon_1\varepsilon'}(u) \widetilde{\otimes} \widehat{L}^+_{\varepsilon'\varepsilon_2}(u),$$
$$\Delta(\mathrm{e}^Q) = \mathrm{e}^Q \widetilde{\otimes} \mathrm{e}^Q,$$
$$\Delta(\mu_l(\widehat{f})) = \mu_l(\widehat{f}) \widetilde{\otimes} 1, \qquad \Delta(\mu_r(\widehat{f})) = 1 \widetilde{\otimes} \mu_r(\widehat{f}).$$

We also define an H-algebra anti-homomorphism $S : U_{q,p} \to U_{q,p}$ by

$$S\big(\widehat{L}^+_{++}\big) = \widehat{L}^+_{--}(u-1), \qquad S\big(\widehat{L}^+_{+-}(u)\big) = -\frac{[P+h+1]}{[P+h]}\widehat{L}^+_{+-}(u-1),$$
$$S\big(\widehat{L}^+_{-+}(u)\big) = -\frac{[P]^*}{[P+1]^*}\widehat{L}^+_{-+}(u-1), \qquad S(\widehat{L}^+_{--}(u)) = \frac{[P+h+1][P]^*}{[P+h][P+1]^*}\widehat{L}^+_{++}(u-1),$$
$$S(\mathrm{e}^Q) = \mathrm{e}^{-Q}, \qquad S(\mu_r(\widehat{f})) = \mu_l(\widehat{f}), \qquad S(\mu_l(\widehat{f})) = \mu_r(\widehat{f}).$$

In fact, one can show that Δ and S preserve the RLL relation (1.2). Moreover, we have the following lemma indicating that ε, Δ and S satisfy the axioms for the counit, the comultiplication and the antipode. Hence the H-algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with (Δ, ε, S) is an H-Hopf algebroid [7–9].

Lemma 2.2. *The maps ε, Δ and S satisfy*

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta,$$
$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta.$$
$$m \circ (\mathrm{id} \otimes S) \circ \Delta(x) = \mu_l(\varepsilon(x)1), \qquad \forall x \in U_{q,p},$$
$$m \circ (S \otimes \mathrm{id}) \circ \Delta(x) = \mu_r(T_\alpha(\varepsilon(x)1)), \qquad \forall x \in (U_{q,p})_{\alpha\beta}.$$

Definition 2.3. *We call the H-Hopf algebroid $(U_{q,p}(\widehat{\mathfrak{sl}}_2), H, M_{H^*}, \mu_l, \mu_r, \Delta, \varepsilon, S)$ the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.*

3. Representations

We consider the dynamical representations, i.e. the representations as H-algebras [7, 8, 12], of the elliptic algebra $U_{q.p}(\widehat{\mathfrak{sl}}_2)$.

3.1. Evaluation representation

We construct the evaluation representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by using the one of $\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)]$. We define the $(l+1)$-dimensional vector space over $\mathbb{F}$ by $V^{(l)} = \bigoplus_{m=0}^{l} \mathbb{F} v^l_m$. Here, v^l_m $(0 \leqslant m \leqslant l)$ denote the weight vectors satisfying $hv^l_m = (l-2m)v^l_m$. Consider the operator $S^{\pm}$ acting on

$V^{(l)}$ by $S^{\pm}v^l_m = v^l_{m\mp1}$, $v^l_m = 0$ for $m < 0, m > l$. In terms of the Drinfeld generators, the evaluation representation $\left(\pi_{l,w}, V^{(l)}_w = V^{(l)}\otimes\mathbb{C}[w,w^{-1}]\right)$ of $\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)]$ is given by [2]

$$\pi_{l,w}(c) = 0, \qquad \pi_{l,w}(d) = 0,$$
$$\pi_{l,w}(a_n) = \frac{w^n}{n}\frac{1}{q-q^{-1}}((q^n+q^{-n})q^{nh} - (q^{(l+1)n}+q^{-(l+1)n})),$$
$$\pi_{l,w}(x^{\pm}(z)) = S^{\pm}\left[\frac{\pm h+l+2}{2}\right]_q \delta\left(q^{h\pm1}\frac{w}{z}\right).$$

Note that $V^{(l)}_w = \bigoplus_{\mu\in\{-l,-l+2,\ldots,l\}} V_\mu$ with V_μ, $\mu = l-2m$ spanned by $v^l_m\otimes w^n (n\in\mathbb{Z})$.

Let us define the H-algebra $\mathcal{D}_{H,V}$ by

$$\mathcal{D}_{H,V} = \bigoplus_{\alpha,\beta\in\bar{H}^*}(\mathcal{D}_{H,V})_{\alpha\beta},$$
$$(\mathcal{D}_{H,V})_{\alpha\beta} = \left\{X\in\mathrm{End}_{\mathbb{C}}V^{(l)}_w \,\middle|\, \begin{array}{c} X(f^*(P)v) = f^*(P-\langle\beta,P\rangle)X(v),\ v\in V^{(l)}_w \\ X(V_\mu)\subseteq V_{\mu+\phi^{-1}(\alpha)-\phi^{-1}(\beta)},\ f^*(P)\in\mathbb{F}\end{array}\right\},$$
$$\mu_l^{\mathcal{D}_{H,V}}(\widehat{f})v = f(P+\mu)v, \qquad \mu_r^{\mathcal{D}_{H,V}}(\widehat{f})v = f^*(P)v$$

for $v\in V_\mu$, then $\widehat{\pi}_{l,w} = \pi_{l,w}\otimes\mathrm{id} : U_{q,p}(\widehat{\mathfrak{sl}}_2) = \mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)]\otimes_{\mathbb{C}}\mathbb{C}[\bar{H}^*]\to\mathcal{D}_{H,V}$ with $e^Q v^l_m = v^l_m$ yields the H-algebra homomorphism. We call $\left(\widehat{\pi}_{l,w}, V^{(l)}_w\right)$ the dynamical evaluation representation. In particular, applying this to definitions 1.2, 1.7 and 1.8, we obtain the following expressions for the images of the $\widehat{L}^+(u)$ operator.

Theorem 3.1.

$$\widehat{\pi}_{l,w}\left(\widehat{L}^+_{++}(u)\right) = -\frac{\left[u-v+\frac{h+1}{2}\right]\left[P-\frac{l-h}{2}\right]\left[P+\frac{l+h+2}{2}\right]}{\varphi_l(u-v)[P][P+h+1]}\,\mathrm{e}^Q,$$
$$\widehat{\pi}_{l,w}\left(\widehat{L}^+_{+-}(u)\right) = -S^-\frac{\left[u-v+\frac{h-1}{2}+P\right]\left[\frac{l-h+2}{2}\right]}{\varphi_l(u-v)[P+h-1]}\,\mathrm{e}^{-Q},$$
$$\widehat{\pi}_{l,w}\left(\widehat{L}^+_{-+}(u)\right) = S^+\frac{\left[u-v-\frac{h+1}{2}-P\right]\left[\frac{l+h+2}{2}\right]}{\varphi_l(u-v)[P]}\,\mathrm{e}^Q,$$
$$\widehat{\pi}_{l,w}(\widehat{L}^+_{--}(u)) = -\frac{\left[u-v-\frac{h-1}{2}\right]}{\varphi_l(u-v)}\,\mathrm{e}^{-Q},$$

where we set $z = q^{2u}$, $w = q^{2v}$ *and*

$$\varphi_l(u) = -z^{-l/2r}\rho^+_{1l}(z,p)^{-1}\left[u+\frac{l+1}{2}\right],$$
$$\rho^+_{kl}(z,p) = q^{kl/2}\frac{\{pq^{k-l+2}z\}\{pq^{-k+l+2}z\}}{\{pq^{k+l+2}z\}\{pq^{-k-l+2}z\}}\frac{\{q^{k+l+2}/z\}\{q^{-k-l+2}/z\}}{\{q^{k-l+2}/z\}\{q^{-k+l+2}/z\}}.$$

The following proposition indicates a consistency of our construction of $\widehat{\pi}_{l,w}$ and the fusion construction of the dynamical R matrices (face-type Boltzmann weights).

Proposition 3.2. *Let us define the matrix elements of* $\widehat{\pi}_{l,w}\left(\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u)\right)$ *by*

$$\widehat{\pi}_{l,w}\left(\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u)\right)v^l_m = \sum_{m'=0}^{l}\left(\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u)\right)_{\mu_{m'}\mu_m}v^l_{m'},$$

where $\mu_m = l-2m$. *Then we have*

$$\left(\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u)\right)_{\mu_{m'}\mu_m} = R^+_{1l}(u-v,P)^{\varepsilon_2\mu_m}_{\varepsilon_1\mu_{m'}}.$$

Here, $R^+_{1l}(u-v,P)$ is the R matrix from (C.17) in [2]. The case $l=1$, $R^+_{11}(u-v,P)$ coincides with the image $(\pi_{1,z}\otimes\pi_{1,w})$ of the universal R matrix $\mathcal{R}^+(\lambda)$ [4] given in (1.3). The case $l>1$, $R^+_{1l}(u-v,P)$ coincides with the R matrix obtained by fusing $R^+_{11}(u-v,P)$ l-times. In particular, the matrix element $R^+_{1l}(u-v,P)^{\varepsilon'\mu'}_{\varepsilon\mu}$ is gauge equivalent to the fusion face weight $W_{l1}(P+\varepsilon',P+\varepsilon'+\mu',P+\mu,P|u-v)$ from (4) in [13].

3.2. Infinite dimensional representation

Let $V(\lambda_l)$ be the level-$k(c=k)$ irreducible highest weight $\mathbb{F}[U_q(\widehat{\mathfrak{sl}}_2)]$-module of highest weight $\lambda_l=(k-l)\Lambda_0+l\Lambda_1$ $(0\leqslant l\leqslant k)$. Here, Λ_i $(i=0,1)$ denote the fundamental weights of $\widehat{\mathfrak{sl}}_2$. We regard $\widehat{V}(\lambda)=\bigoplus_{m\in\mathbb{Z}}V(\lambda)\otimes\mathbb{C}\,\mathrm{e}^{-mQ}$ as the $U_{q,p}(\widehat{\mathfrak{sl}}_2)$-module [2].

We realize $\widehat{V}(\lambda_l)$ by using the Drinfeld generators $a_n(n\in\mathbb{Z}_{\neq0})$ and the q-deformed $\mathbb{Z}_k$-parafermion algebra [1, 2, 14]. Let us define $\alpha_n(n\in\mathbb{Z}_{\neq0})$ by

$$\alpha_n=\begin{cases}a_n & \text{for}\quad n>0\\ \frac{[rn]_q}{[r^*n]_q}q^{k|n|}a_n & \text{for}\quad n<0,\end{cases}$$

with $r^*=r-k$. Then we have

$$[\alpha_m,\alpha_n]=\frac{[2m]_q[km]_q}{m}\frac{[rm]_q}{[r^*m]_q}\delta_{m+n,0}.$$

The q-deformed $\mathbb{Z}_k$-parafermion algebra is an associative algebra over $\mathbb{C}$ generated by $\Psi_{+,\frac{\mu}{k}-n},\Psi_{-,\frac{\mu}{k}-n}(\mu,n\in\mathbb{Z})$. Consider the generating functions (parafermion fields)

$$\Psi(z)\equiv\Psi^+(z)=\sum_{n\in\mathbb{Z}}\Psi_{+,\frac{\mu}{k}-n}z^{-\mu/k+n-1},$$
$$\Psi^\dagger(z)\equiv\Psi^-(z)=\sum_{n\in\mathbb{Z}}\Psi_{-,\frac{\mu}{k}-n}z^{\mu/k+n-1}$$

defined on a weight vector v satisfying $q^hv=q^\mu v$. The parafermion fields $\Psi(z)$ and $\Psi^\dagger(z)$ satisfy

$$\left(\frac{z}{w}\right)^{2/k}\frac{(x^{-2}w/z;x^{2k})_\infty}{(x^{2+2k}w/z;x^{2k})_\infty}\Psi^\pm(z)\Psi^\pm(w)=\left(\frac{w}{z}\right)^{2/k}\frac{(x^{-2}z/w;x^{2k})_\infty}{(x^{2+2k}z/w;x^{2k})_\infty}\Psi^\pm(w)\Psi^\pm(z),$$
$$\left(\frac{z}{w}\right)^{-2/k}\frac{(x^{2+k}w/z;x^{2k})_\infty}{(x^{-2+k}w/z;x^{2k})_\infty}\Psi^\pm(z)\Psi^\mp(w)-\left(\frac{w}{z}\right)^{-2/k}\frac{(x^{2+k}z/w;x^{2k})_\infty}{(x^{-2+k}z/w;x^{2k})_\infty}\Psi^\mp(w)\Psi^\pm(z)$$
$$=\frac{1}{x-x^{-1}}\left(\delta\left(x^k\frac{w}{z}\right)-\delta\left(x^{-k}\frac{w}{z}\right)\right).$$

Theorem 3.3. *[14] By using the irreducible q-$\mathbb{Z}_k$ parafermion module $\mathcal{H}^{PF}_{l,M}$, the level-k irreducible highest weight $U_{q,p}(\widehat{\mathfrak{sl}}_2)$-module $\widehat{V}(\lambda_l)$ is realized as follows:*

$$\widehat{V}(\lambda_l)=\bigoplus_{m\in\mathbb{Z}}\bigoplus_{n\in\mathbb{Z}}\bigoplus_{\substack{M\equiv0\bmod 2k\\(M\equiv l\bmod 2)}}^{2k-1}\widehat{V}(\lambda_l)_{M+2kn+m},$$
$$\widehat{V}(\lambda_l)_{M+2kn+m}=\mathbb{F}[\alpha_{-m}(m\in\mathbb{Z}_{>0})]\otimes\mathcal{H}^{PF}_{l,M}\otimes\mathbb{C}\mathrm{e}^{(M+2kn)\alpha/2}\otimes\mathbb{C}\mathrm{e}^{-mQ}.$$

The action of the elliptic currents on $\widehat{V}(\lambda_l)$ are given by

$$K(u)\mapsto:\exp\left(-\sum_{m\neq0}\frac{[m]_q}{[2m]_q[rm]_q}\alpha_{-m}z^m\right):\mathrm{e}^Qz^{-k(2P-1)/4rr^*+h/2r},$$

$$E(u) \mapsto \Psi(z) : \exp\left(-\sum_{m\neq 0}\frac{1}{[km]_q}\alpha_m z^{-m}\right) : \mathrm{e}^{2Q+\alpha_1} z^{(h+1)/2-(P-1)/r^*},$$

$$F(u) \mapsto \Psi(z)^\dagger : \exp\left(\sum_{m\neq 0}\frac{[r^*m]_q}{[km]_q[rm]_q}\alpha_m z^{-m}\right) : \mathrm{e}^{-\alpha_1} z^{-(h-1)/2+(P+h-1)/r}.$$

Let $(\widehat{\pi}_V, V)$, $(\widehat{\pi}_W, W)$ be two dynamical representations of $U_{q,p}$. We define the tensor product $V\widetilde{\otimes} W$ by

$$V\widetilde{\otimes} W = \bigoplus_{\alpha\in\bar{\mathfrak{h}}^*}(V\widetilde{\otimes} W)_\alpha, \qquad (V\widetilde{\otimes} W)_\alpha = \bigoplus_{\beta\in\bar{\mathfrak{h}}^*} V_\beta \otimes_{M_{H^*}} W_{\alpha-\beta},$$

where $\otimes_{M_{H^*}}$ denotes the usual tensor product modulo the relation

$$f^*(P)v\otimes w = v\otimes f(P+h)w, \tag{3.1}$$

then $(\widehat{\pi}_V \widetilde{\otimes} \widehat{\pi}_W)\circ\Delta : U_{q,p} \to \mathcal{D}_{H,V}\widetilde{\otimes}\mathcal{D}_{H,W}$ is a dynamical representation of $U_{q,p}$ on $V\widetilde{\otimes} W$.

4. Vertex operators

By using the H-Hopf algebroid structure, we define the types I and II vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ as intertwiners of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ modules. Investigating their intertwining relations, we show that they coincide with those obtained in [2] by using the quasi-Hopf algebra structure of $\mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)$ and the isomorphism $U_{q,p}(\widehat{\mathfrak{sl}}_2) \cong \mathcal{B}_{q,\lambda}(\widehat{\mathfrak{sl}}_2)\otimes_{\mathbb{C}}\mathbb{C}[\bar{H}^*]$.

Definition 4.1. *The types I and II vertex operators of spin $n/2$ are the intertwiners of $U_{q,p}$-modules of the form*

$$\widehat{\Phi}(u) : \widehat{V}(\lambda) \to V_z^{(n)}\widetilde{\otimes}\widehat{V}(\nu),$$
$$\widehat{\Psi}^*(u) : \widehat{V}(\lambda)\widetilde{\otimes} V_z^{(n)} \to \widehat{V}(\nu),$$

where $z = q^{2u}$, and $\widehat{V}(\lambda)$ and $\widehat{V}(\nu)$ denote the level-k highest weight $U_{q,p}$-modules of highest weights λ and ν, respectively. They satisfy the intertwining relations with respect to the comultiplication Δ in definition 2.1.

$$\Delta(x)\widehat{\Phi}(u) = \widehat{\Phi}(u)x \qquad \forall x\in U_{q,p}, \tag{4.1}$$

$$x\widehat{\Psi}^*(u) = \widehat{\Psi}^*(u)\Delta(x) \qquad \forall x\in U_{q,p}. \tag{4.2}$$

The physically interesting cases are $n = k$, $\lambda = \lambda_l$, $\nu = \lambda_{k-l}$ for the type I and $n = 1$, $\lambda = \lambda_l$, $\nu = \lambda_{l\pm 1}$ for the type II. See, for example, [14].

Let us define the components of the vertex operators as follows:

$$\widehat{\Phi}\left(v-\frac{1}{2}\right) = \sum_{m=0}^{n} v_m^n\widetilde{\otimes}\Phi_m(v), \tag{4.3}$$

$$\widehat{\Psi}^*\left(v-\frac{c+1}{2}\right)\left(\cdot\,\widetilde{\otimes}\, v_m^n\right) = \Psi_m^*(v). \tag{4.4}$$

Theorem 4.2. *The vertex operators satisfy the following linear equations:*

$$\widehat{\Phi}(u)\widehat{L}^+(v) = R_{1n}^{+(12)}(v-u, P+h)\widehat{L}^+(v)\widehat{\Phi}(u), \tag{4.5}$$

$$\widehat{L}^+(v)\widehat{\Psi}^*(u) = \widehat{\Psi}^*(u)\widehat{L}^+(v)R^{+*(13)}_{1n}(v-u, P-h^{(1)}-h^{(3)}). \tag{4.6}$$

Relation (4.5) should be understood on $V^{(1)}_w \widetilde{\otimes} \widehat{V}(\lambda)$, *whereas (4.6) on* $V^{(1)}_w \widetilde{\otimes} \widehat{V}(\lambda) \widetilde{\otimes} V^{(n)}_z$.

Proof. Applying Δ in definition 2.1 and noting proposition 3.2, we obtain from (4.1)

$$\begin{aligned}\widehat{\Phi}(u)\widehat{L}^+_{\varepsilon_1\varepsilon_2}(v) &= \Delta\big(\widehat{L}^+_{\varepsilon_1\varepsilon_2}(v)\big)\widehat{\Phi}(u)\\ &= \sum_{m=0}^{n}\sum_{\varepsilon}\widehat{L}^+_{\varepsilon_1\varepsilon}(v)v^n_m \widetilde{\otimes} \widehat{L}^+_{\varepsilon\varepsilon_2}(v)\Phi_m(u)\\ &= \sum_{m=0}^{n}\sum_{\varepsilon}\sum_{m'=0}^{n} R^+_{1n}(v-u, P)^{\varepsilon\mu_m}_{\varepsilon_1\mu_{m'}} v^n_{m'} \widetilde{\otimes} \widehat{L}^+_{\varepsilon\varepsilon_2}(v)\Phi_m(u)\\ &= \sum_{m'=0}^{n} v^n_{m'} \widetilde{\otimes} \sum_{m=0}^{n}\sum_{\varepsilon} R^+_{1n}(v-u, P+h)^{\varepsilon\mu_m}_{\varepsilon_1\mu_{m'}} \widehat{L}^+_{\varepsilon\varepsilon_2}(v)\Phi_m(u),\end{aligned}$$

where $\mu_m = n-2m$, etc. In the last equality we used (3.1). Similarly, for the type II, from (4.2), we obtain

$$\begin{aligned}\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u)\Psi^*_m\left(v+\frac{1}{2}\right) &= \widehat{\Psi}^*\left(v+\frac{1}{2}\right)\Delta\big(\widehat{L}^+_{\varepsilon_1\varepsilon_2}(u)\big)\big(\cdot \widetilde{\otimes} v^n_m\big)\\ &= \sum_{\varepsilon}\sum_{m'}\widehat{\Psi}^*\left(v+\frac{1}{2}\right)\big(\widehat{L}^+_{\varepsilon_1\varepsilon}(u) \widetilde{\otimes} R^+_{1n}(u-v, P)^{\varepsilon_2\mu_m}_{\varepsilon\mu_{m'}} v^n_{m'}\big)\\ &= \sum_{\varepsilon}\sum_{m'}\widehat{\Psi}^*\left(v+\frac{1}{2}\right)\big(R^{+*}_{1n}(u-v, P-\mu_{m'})^{\varepsilon_2\mu_m}_{\varepsilon\mu_{m'}} \widehat{L}^+_{\varepsilon_1\varepsilon}(u) \widetilde{\otimes} v^n_{m'}\big)\\ &= \sum_{\varepsilon}\Psi^*_{m'}\left(v+\frac{1}{2}\right)R^{+*}_{1n}(u-v, P-\mu_{m'})^{\varepsilon_2\mu_m}_{\varepsilon\mu_{m'}} \widehat{L}^+_{\varepsilon_1\varepsilon}(u)\\ &= \sum_{\varepsilon}\Psi^*_{m'}\left(v+\frac{1}{2}\right)\widehat{L}^+_{\varepsilon_1\varepsilon}(u)R^{+*}_{1n}(u-v, P-\mu_{m'}-\varepsilon)^{\varepsilon_2\mu_m}_{\varepsilon\mu_{m'}}.\end{aligned}$$

Here in the third equality, we used relation (3.1). Note also $\varepsilon + \mu_{m'} = \varepsilon_2 + \mu_m$. □

Equations (4.5) and (4.6) coincide with (5.3) and (5.4) in [2], respectively. Note that the comultiplication used in [2] corresponds to the opposite one of Δ here. Under certain analyticity conditions, these equations determine the vertex operators uniquely up to normalization.

Acknowledgments

The author would like to thank Michio Jimbo, Anatol Kirillov, Atsushi Nakayashiki, Masatoshi Noumi and Hjalmar Rosengren for stimulating discussions and valuable suggestions. This work is supported by the Grant-in-Aid for Scientific Research (C)19540033, JSPS Japan.

References

[1] Konno H 1998 An elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and the fusion RSOS models *Commun. Math. Phys.* **195** 373–403
[2] Jimbo M, Konno H, Odake S and Shiraishi J 1999 Elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$: Drinfeld currents and vertex operators *Commun. Math. Phys.* **199** 605–47
[3] Drinfeld V G 1988 A new realization of yangians and quantized affine algebras *Sov. Math. Dokl.* **36** 212–6

[4] Jimbo M, Konno H, Odake S and Shiraishi J 1999 Quasi-Hopf twistors for elliptic quantum groups *Transformation Groups* **4** 303–27

[5] Felder G 1995 Elliptic quantum groups *Proc. ICMP Paris-1994* pp 211–8

[6] Enriquez B and Felder G 1998 Elliptic quantum groups $E_{\tau,\eta}(\mathfrak{sl}_2)$ and quasi-Hopf algebra *Commun. Math. Phys.* **195** 651–89

[7] Etingof P and Varchenko A 1998 Solutions of the quantum dynamical Yang–Baxter equation and dynamical quantum groups *Commun. Math. Phys.* **196** 591–640

[8] Etingof P and Varchenko A 1999 Exchange dynamical quantum groups *Commun. Math. Phys.* **205** 19–52

[9] Koelink E and Rosengren H 2001 Harmonic analysis on the $SU(2)$ dynamical quantum group *Acta Appl. Math.* **69** 163–220

[10] Koelink E, van Norden Y and Rosengren H 2004 Elliptic $U(2)$ quantum group and elliptic hypergeometric series *Commun. Math. Phys.* **245** 519–37

[11] Konno H Elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, Hopf algebroid structure and elliptic hypergeometric series (in preparation)

[12] Felder G and Varchenko A 1996 On representations of the elliptic quantum groups $E_{\tau,\eta}(\mathfrak{sl}_2)$ *Commun. Math. Phys.* **181** 741–61

[13] Date E, Jimbo M, Miwa T and Okado M 1986 Fusion of the eight-vertex SOS model *Lett. Math. Phys.* **12** 209–15

[14] Kojima T, Konno H and Weston R 2005 The vertex-face correspondence and correlation functions of the fusion eight-vertex models: I. The general formalism *Nucl. Phys.* B **720** 348–98